

Relic Gravitational Waves

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Abstract. The next generation of gravitational wave detectors holds out the prospect of detecting a stochastic gravitational wave background generated in the very early universe. In this article, we review the various cosmological processes which can lead to such a background, including quantum fluctuations during inflation, bubble collisions in a first-order phase transition and the decay of a network of cosmic strings. We conclude that signals from strongly first-order phase transitions, possibly at the end of inflation, and networks of local cosmic strings are within the sensitivity of proposed detectors. However, backgrounds from standard slow-roll inflation and the electroweak phase transition are too weak.

1. Introduction

An outstanding achievement of modern cosmology has been the detection of anisotropies in the cosmic microwave background [1]. These anisotropies provide a snapshot of the universe about 400,000 years after the Hot Big Bang, just as the universe became transparent to electromagnetic radiation. Despite this observational triumph, as yet it has been insufficient to differentiate between competing paradigms for galaxy formation, nor has it shed much light on cosmological processes taking place before photon decoupling. There are, however, other types of radiation, such as gravitational radiation, which penetrate through this electromagnetic surface of last scattering, travelling virtually unaffected since their emission, even when this occurred in the first few fractions of a second. Of course, for gravitons this remarkable transparency is due to their very weak interactions with ordinary matter which, in turn, makes them difficult to observe. However, pioneering experiments have been proposed which could detect stochastic backgrounds of gravitational waves generated in the early universe over a range of frequencies.

Gravitational radiation detected with a particular frequency today would have been created at a characteristic time (if generated by classical, causal mechanisms). Assuming

that the radiation is emitted at a time t_e before equal matter-radiation ($t_e < t_{\text{eq}} \sim 40,000$ years) and that it is created with a wavelength comparable to the horizon $\lambda(t_e) \sim t_e$, then its frequency today is given by $f \sim z_{\text{eq}}^{-1}(t_{\text{eq}}t_e)^{-1/2}$ where the redshift $z_{\text{eq}} \sim 2.3 \times 10^4 \Omega_0 h^2$ and ‘little h ’ is the rescaled Hubble constant lying in the range $0.4 \leq h \leq 0.9$. The quantity that is measured by a particular experiment is the dimensionless amplitude

$$h_c(f) = 1.3 \times 10^{-20} \sqrt{\Omega_g(f) h^2} \left(\frac{100 \text{Hz}}{f} \right), \quad \Omega_g(f) = \frac{f}{\rho_c} \frac{\partial \rho_g}{\partial f}, \quad (1)$$

where $\Omega_g(f)$ is the gravitational radiation density contribution to the universe’s density per octave at a frequency f , and ρ_c is the critical energy density. Clearly, from (1) gravitational radiation with higher frequencies will be more difficult to detect.

There are currently four frequency bands available for studying gravitational radiation [2]. First, there are the microwave background anisotropies created by tensor modes which may have been detected by the COBE-DMR experiment. In inflationary scenarios, the precise signal ratio due to tensor versus scalar modes is sensitively model-dependent, but it could be as high as 50%. Assuming that the entire COBE signal is due to gravitational waves implies a weak upper bound of $\Omega_g < 7 \times 10^{-11}$ at a frequency of $3 \times 10^{-17} h$ Hz. A second upper limit on Ω_g comes from pulsars, since a stochastic background would lead to timing noise in measurements of the incoming periodic electromagnetic signal. A recent limit is $\Omega_g h^2 < 6 \times 10^{-8}$ at 4×10^{-9} Hz [3], though the statistical veracity of this result has been questioned with re-analyses of the same data suggesting both stronger and weaker bounds [4, 5]. Thirdly, proposed ground based detectors will study frequencies around 100Hz with a maximum sensitivity of about $\Omega_g h^2 \approx 10^{-7}$ for the first generation of LIGO detectors and $\Omega_g h^2 \approx 10^{-10}$ for the next generation LIGO and VIRGO detectors. Finally, the proposed space-based interferometer LISA will have a sensitivity of $\Omega_g h^2 \approx 10^{-10}$ at about 10^{-3} Hz.

2. Slow-roll inflation

The inflationary paradigm [6, 7, 8] is strongly motivated because it resolves a number of shortcomings of the standard Hot Big Bang cosmology, including the horizon, flatness and monopole problems. However, possibly the most significant testable prediction of inflationary scenarios is a nearly scale-invariant spectrum of adiabatic density perturbations produced through quantum mechanical effects when the very early universe is dominated by the potential energy of some scalar inflaton field ϕ . An important by-product of this process is the excitation of tensor modes resulting in a background of gravitational waves. The spectra for scalar and tensor modes created in a slow-roll inflationary model are the following,

$$\mathcal{P}_S(\omega) = \frac{1}{\pi \epsilon(\phi_\omega)} \left(\frac{H(\phi_\omega)}{m_{\text{pl}}} \right)^2, \quad \mathcal{P}_T(\omega) = \frac{16}{\pi} \left(\frac{H(\phi_\omega)}{m_{\text{pl}}} \right)^2, \quad (2)$$

where $H(\phi)$ is the Hubble parameter during inflation, ϕ_ω is the value of the inflaton field when the mode with comoving wavenumber $\omega = 2\pi f$ leaves the Hubble radius H^{-1} , and the slow-roll parameter is

$$\epsilon(\phi) = \frac{m_{\text{pl}}^2}{4\pi} \left(\frac{H'(\phi)}{H(\phi)} \right)^2, \quad (3)$$

with the prime ' denoting differentiation with respect ϕ . Here, $\epsilon(\phi)$ is related to the speed at which ϕ rolls down the potential $V(\phi)$ (less than unity during inflation). Once a particular mode is created, it is driven outside the Hubble radius (or 'horizon') by the rapid expansion and it effectively 'freezes' until it returns inside the horizon during the subsequent radiation and matter dominated eras. Lengthscales corresponding to the size of the observed universe went outside the horizon early when the inflaton field had a value denoted by ϕ_{60} ; after this there were about 60 e-foldings of expansion before the end of inflation when $\epsilon(\phi_{\text{end}}) = 1$.

For a specific inflaton potential $V(\phi)$, the dynamics of the Hubble parameter in the Hamilton-Jacobi formalism is given by [9]

$$H'^2 - \frac{12\pi}{m_{\text{pl}}^2} H^2 = -\frac{32\pi^2}{m_{\text{pl}}^2} V(\phi), \quad \dot{\phi} = -\frac{m_{\text{pl}}^2}{4\pi} H'. \quad (4)$$

The Hubble parameter monotonically decreases during slow-roll inflation, so from (2) the largest contribution to the gravitational wave spectrum is due to modes that were driven outside the horizon early in inflation and have just come back inside the horizon at the present day. These scales correspond to those probed by the COBE-DMR experiment and the observed anisotropies can be used with (2) to normalize $H_{60} \equiv H(\phi_{60})$. Given this initial condition, (4) can be used to calculate the Hubble parameter during inflation and hence the resulting spectrum of gravitational waves. Paradoxically, as we shall see, the larger the contribution of a model to the COBE signal, the smaller the signal will be in the frequency bands corresponding to LISA and LIGO/VIRGO; the Hubble parameter must decrease more rapidly in order to maintain the condition that there be 60 e-foldings before the end of inflation [10].

In order to illustrate this point, we shall calculate the spectrum of gravitational radiation produced by an inflationary model similar to polynomial chaotic inflation, but slightly modified to make analytic calculations tractable [10]. Using a simple method to obtain the COBE normalization [11], the potential for this model is given by

$$V(\phi) = 6.7 \times 10^{-10} m_{\text{pl}}^4 \left(1 + \frac{240}{\alpha} \right)^{-1} \left(\frac{\phi}{\phi_{60}} \right)^\alpha \left(1 - \frac{\alpha^2 m_{\text{pl}}^2}{48\pi\phi^2} \right), \quad (5)$$

where $\phi_{60} = -(\alpha m_{\text{pl}}/4\sqrt{\pi})(1 + 240/\alpha)^{1/2}$ and we require $\alpha > 1$. For this model, the slow-roll and Hubble parameters are given by

$$\epsilon(\phi) = \frac{\alpha^2 m_{\text{pl}}}{16\pi\phi^2}, \quad \frac{H(\phi)}{m_{\text{pl}}} = 7.5 \times 10^{-5} \left(1 + \frac{240}{\alpha} \right)^{-1/2} \left(\frac{\phi}{\phi_{60}} \right)^{\alpha/2}. \quad (6)$$

For $\omega > \omega_{\text{eq}}$, substituting into (2) and appropriately redshifting yields

$$\Omega_g(\omega)h^2 \approx 4.6 \times 10^{-14} \left(1 + \frac{240}{\alpha}\right)^{-1} \left[1 - \frac{4}{240 + \alpha} \log\left(\frac{\omega}{a_0 H_0}\right)\right]^{\alpha/2}. \quad (7)$$

For $\omega < \omega_{\text{eq}}$, that is, scales which come inside the horizon later during the matter era, we have less redshifting and (7) must be multiplied by an additional factor $(\omega_{\text{eq}}/\omega)^2$.

To delineate the predictions of different slow-roll inflation models, one can use different values of α ; the obvious limiting cases being $\alpha = 2$, the simplest chaotic model, and $\alpha = 80$ (say) for which almost all the COBE signal is due to gravitational waves (this case corresponds closely to power law inflation with an exponential potential). The results of these cases are plotted in figure 1. The $\alpha = 2$ model has a low contribution to COBE but the spectrum for $\omega > \omega_{\text{eq}}$ decreases slowly down to an amplitude of $\Omega_g h^2 \approx 10^{-16}$ at a few Hz. In contrast, the model with $\alpha = 80$ has a much larger contribution to COBE, but has a very small amplitude for scales $\omega > \omega_{\text{eq}}$; in the LISA frequency band this has already fallen to $\Omega_g h^2 \approx 10^{-24}$, eight orders of magnitude smaller than for $\alpha = 2$. Nonetheless, both these models produce signals well below the sensitivity of proposed interferometers by several orders of magnitude.

The largest possible COBE normalization provides a weak upper limit on the amplitude of gravitational waves for all scales $\omega > \omega_{\text{eq}}$ which must be below the level $\Omega_g h^2 \approx 10^{-14}$. It may be possible to carefully construct a slow-roll potential which would almost achieve this limit in the LISA and LIGO/VIRGO frequency bands, but undoubtedly the potential would look rather contrived. In any case, it is clear that the prospects for a detectable slow-roll signal are bleak, so we shall have to consider less orthodox inflationary models. In the next section, we consider extended and hybrid inflation, but there are also more speculative superstring-inspired models which can give a larger gravitational wave background (for example see [12]).

3. First-order phase transitions

Cosmological phase transitions in which symmetries are spontaneously broken are now an integral part of modern cosmology. In a first-order phase transition the field becomes trapped in a metastable local minimum of the potential—the false vacuum. The transition to the true vacuum takes place by the nucleation of bubbles and their subsequent rapid growth. When these vacuum bubbles collide copious amounts of gravitational radiation can be emitted, particularly if the relative velocity of the walls is relativistic as in strongly first-order phase transitions. It has been estimated that the maximum contribution to the gravitational wave background from such a transition would be [13]

$$\Omega_g(f_{\text{max}})h^2 \approx 10^{-6} \left(\frac{H_*}{\beta}\right)^2 \left(\frac{100}{\mathcal{N}_*}\right)^{1/3}, \quad (8)$$

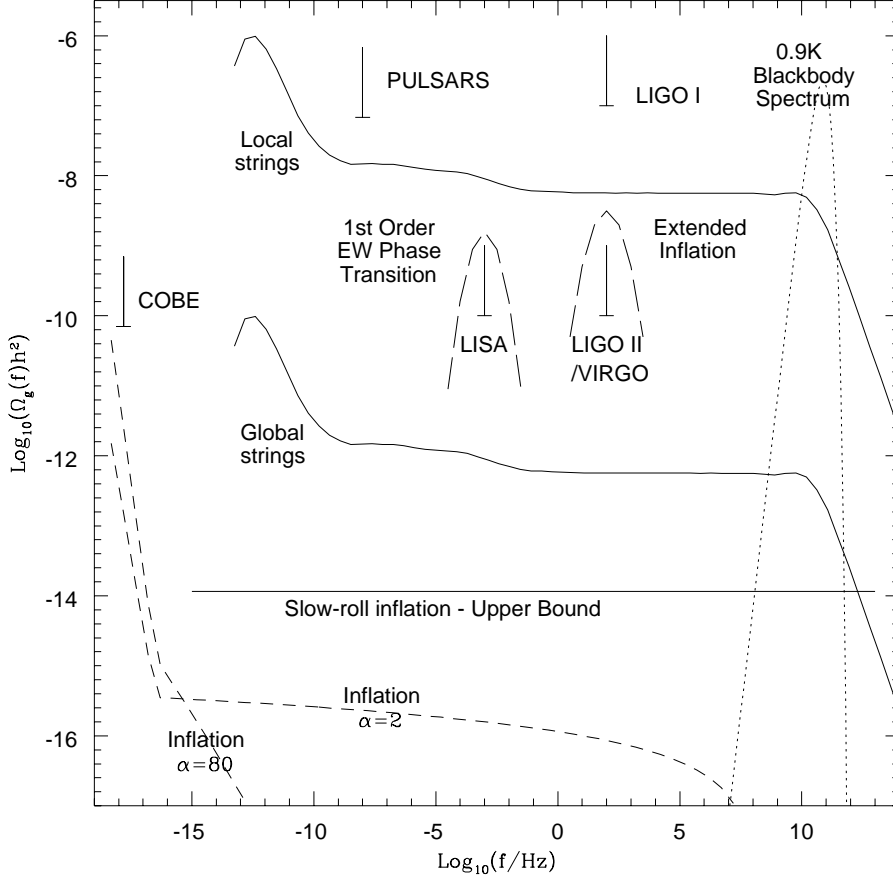


Figure 1. Summary of the potential cosmological sources of a stochastic gravitational radiation background, including inflationary models, first-order phase transitions and cosmic strings, as well as a primordial 0.9K black-body graviton spectrum (the analogue of the black-body photon radiation). Also plotted are the relevant constraints from the COBE measurements, pulsar timings, and the sensitivities of the proposed interferometers. Notice that local cosmic strings and strongly first-order phase transitions may produce detectable backgrounds, in contrast to standard slow-roll inflation models.

at a frequency of

$$f_{\max} \approx 3 \times 10^{-8} \text{Hz} \left(\frac{\beta}{H_*} \right) \left(\frac{\mathcal{N}_*}{100} \right)^{1/6} \left(\frac{T_*}{1 \text{GeV}} \right), \quad (9)$$

where $\Gamma = \Gamma_0 \exp(\beta t)$ is the nucleation rate of bubbles, T_* is the temperature of the phase transition, H_* is the relevant Hubble parameter and \mathcal{N}_* is the number of relativistic degrees of freedom. Assuming that the electroweak phase transition is strongly first-order implies $\Omega_g h^2 \approx 10^{-9}$ at a frequency of $f_{\max} \approx 10^{-3}$, inside the LISA band with a

detectable amplitude, as illustrated in figure 1. However, the minimal standard model is currently believed to have only a weakly first-order transition, if not second-order, with the bubble walls reaching velocities well below the speed of light. In this case, $\Omega_g h^2$ has been estimated to be around 10^{-22} which is well outside LISA's sensitivity [14]. Nevertheless, there are a number of well-motivated extensions to the standard model (like supersymmetry) which could entail symmetry-breaking just above the electroweak scale; if strongly first-order, such phase transitions would create a distinctive and detectable LISA signal. Fortunately from this point of view, LISA has a very interesting frequency response range.

While the prospects for detecting gravitational waves from slow-roll inflation seem poor, there are a number of inflationary models which do not end in the standard reheating scenario. These include extended inflation [15] and hybrid inflation [16] which exit through a phase transition; this provides an extra, potentially more powerful source of gravitational waves if this final phase transition is strongly first-order [17]. The radiation could be detected by the advanced LIGO and VIRGO detectors if the re-heat temperature of the universe was close to 10^8 GeV or 10^9 GeV —scales which are far from ruled out by current observations. We have included a sample spectra for such a model in figure 1.

4. Cosmic string networks

Cosmic strings are line-like topological defects which may form during phase transitions in the early universe (see, for example, [18]). Due to the large string mass per unit length (typically about $10^{22} \text{ g cm}^{-1}$ for GUT scale strings), they may have acted as the initial seeds for the formation of large-scale structure [19, 20]. In general, a network of strings will evolve towards a self-similar scaling solution by the production of loops and the subsequent emission of radiation into a preferred channel. For local strings, this channel is gravitational radiation, thus creating a stochastic background of radiation over frequencies from 10^{10} Hz to 10^{-12} Hz .

Scale-invariant string evolution implies that the density of the string network $\rho_\infty = \mu\zeta/t^2$ remains constant relative to the background density. Statistically, the network has the same properties at any two times, except for a universal scaling with respect to the growing horizon size, that is, there exists a characteristic length scale $L = \zeta^{-1/2}t$. This scaling regime has been shown to exist numerically [21, 22] and various parameters have been estimated, notably $\zeta \approx 13$ during the radiation era.

The radiative dynamics of the strings is a crucial factor determining the precise contributions at various frequencies. The power radiated by a string loop is normally

written as a sum over loop harmonics [23], that is,

$$P = \sum_{n=1}^{n_*} P_n = \Gamma G \mu^2 \quad (10)$$

where $\langle \Gamma \rangle \approx 65$ [24] is a constant parametrized solely by the particular loop trajectory and not its length, μ is the string mass per unit length and n_* is some cut-off introduced by the effects of radiation backreaction [25] (see also [26]).

Assuming that \mathcal{N} is constant and for reasonable values of n_* , one can deduce that the spectrum of gravitational radiation due to loops created during the radiation era is independent of frequency and only very weakly dependent on the loop radiation spectrum. The amplitude of this flat spectrum is given by

$$\Omega_g(\omega) \approx \frac{256\pi G \mu}{3} \Omega_r F\left(\frac{\alpha}{\Gamma G \mu}\right), \quad (11)$$

where $F(x) = ((1+x)^{3/2} - 1)/x$, $\Omega_r = \rho_r/\rho_c$, $\rho_r = 3/32\pi G t^2$ and α is the constant loop production size with respect to the horizon. In contrast, the radiation background today due to loops produced in the matter era is highly dependent on the loop radiation spectrum. Assuming that $P_n \propto n^{-q}$, one can deduce that $\Omega_g(\omega) \propto \omega^{1-q} \rightarrow \log_{10}[\Omega_g(\omega)] = A + (1-q) \log_{10}[\omega]$. Therefore, the amount of radiation produced in the matter era which feeds into higher frequencies is extremely sensitive to q and n_* . It is precisely these frequencies that are relevant for the pulsar timing experiment [27].

One can investigate these effects more fully as in ref. [25] using a numerical algorithm which calculates the spectrum of radiation for given spectra and cosmological parameters [27]. The parameters that were used are $\Omega_0 = 1$, $h = 0.5$ and the electroweak model particle spectrum with three lepton generations. It was found that the contribution to the pulsar timing frequency was the same for all $q \geq 2$ irrespective of the value of n_* , while for $q < 2$ it was sensitive to the value of n_* . In particular, for $q = 4/3$ and $n_* = \infty$ the amplitude of the spectrum will be in conflict with the constraint from pulsar timings for the string energy density $G\mu \approx 10^{-6}$ normalised to COBE [28]. However, if $q = 4/3$ and $n_* < 1000$ then the contribution to the pulsar timing frequency will be the same as for $q = 2$ and $n_* = \infty$. The spectrum of radiation emitted by loops is likely to be cut off by the effects of backreaction as shown numerically for Goldstone boson radiation from global cosmic strings [26], therefore it seems sensible to exclude the $q = 4/3$, large n_* spectra on physical grounds (as we discuss in ref. [25]). Using well-motivated values of q and n_* , one can use the pulsar timing constraint to deduce that $G\mu < 5.4(\pm 0.8) \times 10^{-6}$ [25]. Using $\Omega_0 < 1$ also reduces the contribution to the pulsar timing frequency from matter era loops due to curvature domination at $t_c \approx \Omega_0 t_0$, though we expect this effect to be small.

The contribution from cosmic strings to the LISA and LIGO/VIRGO frequency ranges is likely to be slightly lower than that relevant for pulsar timing because particle

mass thresholds cause entropy transfers which reduce the relative density contribution of any pre-existing decoupled radiation. The cosmic string spectrum shown in figure 1 illustrates this effect with a gradual rise in the otherwise flat spectra at frequencies around 10^{-4}Hz . This particular rise is caused by particle annihilation near the QCD and electroweak phase transitions. If the particle physics model has more degrees of freedom at higher energies there could be other steps related to other phase transitions. However, the dependence on \mathcal{N} is reasonably weak and therefore we can conservatively estimate that $\Omega_{\text{g}}h^2 > 2.0 \times 10^{-9}$ at 10^{-3}Hz and $\Omega_{\text{g}}h^2 > 1.0 \times 10^{-9}$ at 100Hz for $G\mu = 1.0 \times 10^{-6}$. We note also that very precise determinations of the stochastic background from cosmic strings at different frequencies would measure \mathcal{N} in different cosmological epochs, providing fascinating insight into the particle content of the early universe (at times as early as 10^{-25}sec for LIGO and VIRGO).

Finally, we should note that there are other types of cosmic strings which do not produce a large background of gravitational radiation, such as those formed when global symmetries are broken. The dynamics of global strings is dominated by the production of Goldstone bosons and a simple calculation suggests that gravitational waves will be suppressed by approximately four orders of magnitude; the global string spectrum is also illustrated in figure 1.

5. Conclusion

Gravitational waves potentially provide a unique and unparalleled probe of the very early universe. The proposed interferometers, both terrestrial and space-based, could rule out or severely constrain a number of viable theoretical models, most notably the cosmic string scenario. On the other hand, the detection of a primordial background of gravitational waves would have a profound impact on our understanding of high energy physics and cosmology, providing an unprecedented view of the earliest moments after the creation of the Universe.

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